

NONEXISTENCE OF SOLUTIONS FOR THE 1 LAPLACIAN WITH L^N DATA

LUIGI ORSINA AND AUGUSTO C. PONCE

ABSTRACT. We prove that the 1 Laplacian equation $-\operatorname{div} \frac{\nabla u}{|\nabla u|} = f$ in an open set $\Omega \subset \mathbb{R}^N$ cannot have a solution in $W_0^{1,1}(\Omega)$ for any datum $f \in L^N(\Omega)$.

1. MAIN RESULT

Let $\Omega \subset \mathbb{R}^N$ be an open set. The goal of this paper is to give a proof of the following result:

Theorem 1.1. *There exists no function $u \in W_0^{1,1}(\Omega)$ such that $\nabla u \neq 0$ a.e. in Ω and*

$$\operatorname{div} \frac{\nabla u}{|\nabla u|} \in L^N(\Omega).$$

The equation $-\operatorname{div} \frac{\nabla u}{|\nabla u|} = f$ is the counterpart for $p = 1$ of the p Laplacian equation

$$-\Delta_p u = -\operatorname{div} (|\nabla u|^{p-2} \nabla u) = f.$$

For every $1 < p < +\infty$, the p Laplacian gives a continuous bijection from $W_0^{1,p}(\Omega)$ into the dual space $(W_0^{1,p}(\Omega))'$; see [6, Théorème 2].

Every function $u \in W_0^{1,1}(\Omega)$ such that $\nabla u \neq 0$ a.e. in Ω has a 1 Laplacian

$$\Delta_1 u = \operatorname{div} \frac{\nabla u}{|\nabla u|}$$

in the sense of distributions: for every $\varphi \in C_c^\infty(\Omega)$,

$$\langle \Delta_1 u, \varphi \rangle = - \int_{\Omega} \frac{\nabla u}{|\nabla u|} \cdot \nabla \varphi.$$

Since $\frac{\nabla u}{|\nabla u|} \in L^\infty(\Omega)$, we have $\Delta_1 u \in (W_0^{1,1}(\Omega))'$. Moreover, by the Gagliardo-Nirenberg-Sobolev imbedding, $L^N(\Omega) \subset (W_0^{1,1}(\Omega))'$. Hence, according to Theorem 1.1, for every $u \in W_0^{1,1}(\Omega)$ such that $\nabla u \neq 0$ a.e. in Ω ,

$$\Delta_1 u \in (W_0^{1,1}(\Omega))' \setminus L^N(\Omega).$$

We can get a flavor of this lack of surjectivity for the 1 Laplacian by looking at what happens to a smooth function:

2010 *Mathematics Subject Classification.* Primary 35J70; Secondary 35J25, 35J62, 35J92.

Key words and phrases. 1 Laplacian, degenerate elliptic equations, nonlinear elliptic equation, nonexistence of solution.

Example 1. Let $B_1(0)$ be the unit ball in \mathbb{R}^N and let $u : B_1(0) \rightarrow \mathbb{R}$ be the function defined by $u(x) = 1 - |x|^2$. In the sense of distributions we have for $N = 1$,

$$-\operatorname{div} \frac{\nabla u}{|\nabla u|} = 2\delta_0,$$

while for $N \geq 2$,

$$-\operatorname{div} \frac{\nabla u}{|\nabla u|} = \frac{N-1}{|x|}.$$

Thus, in all cases, $\operatorname{div} \frac{\nabla u}{|\nabla u|} \notin L^N(\Omega)$.

Theorem 1.1 clarifies the degenerate limit behavior of solutions of the p Laplacian equation as p tends to 1 that has been studied by several authors, starting with a pioneering work of Kawohl [4, 8–10]. There have been some attempts to give a meaning to a renormalized solution to equations involving the 1 Laplacian somewhat motivated by these works [1–3, 5, 7]. In general such renormalized solutions merely belong to $BV(\Omega)$ or do not satisfy the Dirichlet boundary condition. In [7, Remark 3.10] there are examples of renormalized solutions for the Dirichlet problem

$$\begin{cases} -\Delta_1 v + v = h & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0), \end{cases}$$

for various choices of $h \in L^\infty(B_R(0))$. For instance, for every $N < r \leq R$, the function $u_r = (1 - N/r)\chi_{B_r(0)}$ is a renormalized solution of this problem with datum $h_r = \chi_{B_r(0)}$. Note that if $r < R$ then u_r is a BV function, while if $r = R$ then u_r is a $W^{1,1}$ function which does not vanish on the boundary.

In Theorem 1.1, the assumption $\nabla u \neq 0$ a.e. in Ω cannot be relaxed by assuming that there exists a vector field $Z \in L^\infty(\Omega; \mathbb{R}^N)$ such that

$$(1.1) \quad \nabla u \cdot Z = |\nabla u| \quad \text{a.e. in } \Omega,$$

in which case Z would play the role of $\frac{\nabla u}{|\nabla u|}$:

Example 2. For any $0 < r < 1$, let $u_r : B_1(0) \rightarrow \mathbb{R}$ be the function defined by

$$u_r(x) = \begin{cases} 1 - |x|^2 & \text{if } |x| \geq r, \\ 1 - r^2 & \text{if } |x| < r. \end{cases}$$

Then, $u_r \in W_0^{1,1}(B_1(0))$. If $Z_r : B_1(0) \rightarrow \mathbb{R}^N$ is any smooth extension of the function $x \in B_1(0) \setminus B_r(0) \mapsto -\frac{x}{|x|} \in \mathbb{R}^N$, then u_r and Z_r satisfy (1.1) and $\operatorname{div} Z_r \in L^\infty(B_1(0))$.

2. PROOF OF THEOREM 1.1

Assume by contradiction that there exists a function u satisfying the assumptions of the theorem. Let $f \in L^N(\Omega)$ be such that for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} \frac{\nabla u}{|\nabla u|} \cdot \nabla \varphi = \int_{\Omega} f \varphi.$$

Note that $\frac{\nabla u}{|\nabla u|} \in L^\infty(\Omega)$ and by the Gagliardo-Nirenberg-Sobolev imbedding, $u \in L^{\frac{N}{N-1}}(\Omega)$. By density of $C_c^\infty(\Omega)$ in $W_0^{1,1}(\Omega)$, we deduce that for every $v \in W_0^{1,1}(\Omega)$,

$$(2.1) \quad \int_{\Omega} \frac{\nabla u}{|\nabla u|} \cdot \nabla v = \int_{\Omega} f v.$$

We show that $u \leq 0$ a.e. in Ω . For this purpose, for every $\kappa > 0$ let $G_\kappa : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$G_\kappa(t) = \begin{cases} 0 & \text{if } t \leq \kappa, \\ t - \kappa & \text{if } t > \kappa. \end{cases}$$

Since $u \in W_0^{1,1}(\Omega)$, we have $G_\kappa(u) \in W_0^{1,1}(\Omega)$. Hence,

$$\frac{\nabla u}{|\nabla u|} \cdot \nabla G_\kappa(u) = G'_\kappa(u) |\nabla u| = |\nabla G_\kappa(u)|.$$

Applying identity (2.1) with test function $G_\kappa(u)$, we get

$$\int_{\Omega} |\nabla G_\kappa(u)| = \int_{\Omega} f G_\kappa(u).$$

Since G_κ vanishes on the interval $(-\infty, \kappa]$, by Hölder's inequality we have

$$\int_{\Omega} f G_\kappa(u) = \int_{\{u > \kappa\}} f G_\kappa(u) \leq \|f\|_{L^N(\{u > \kappa\})} \|G_\kappa(u)\|_{L^{\frac{N}{N-1}}(\Omega)}.$$

Thus,

$$\int_{\Omega} |\nabla G_\kappa(u)| \leq \|f\|_{L^N(\{u > \kappa\})} \|G_\kappa(u)\|_{L^{\frac{N}{N-1}}(\Omega)}.$$

By the Gagliardo-Nirenberg-Sobolev inequality,

$$\|G_\kappa(u)\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C \int_{\Omega} |\nabla G_\kappa(u)|,$$

for some constant $C > 0$ depending only on the dimension N . Hence,

$$(2.2) \quad (1 - C\|f\|_{L^N(\{u > \kappa\})}) \|G_\kappa(u)\|_{L^{\frac{N}{N-1}}(\Omega)} \leq 0.$$

Assume by contradiction that $u > 0$ a.e. on a set of positive Lebesgue measure and let $T = \|u^+\|_{L^\infty(\Omega)}$. We have

$$\lim_{\kappa \nearrow T} \|f\|_{L^N(\{u > \kappa\})} = \|f\|_{L^N(\{u = T\})}.$$

We observe that the set $\{u = T\}$ has zero Lebesgue measure. This is indeed the case when $T = +\infty$ since u is finite a.e. When $T < +\infty$, we observe that $\nabla u = 0$ a.e. on the level set $\{u = T\}$; since by assumption $\nabla u \neq 0$ a.e. in Ω , the set $\{u = T\}$ must have zero Lebesgue measure.

This implies that

$$\lim_{\kappa \nearrow T} \|f\|_{L^N(\{u > \kappa\})} = \|f\|_{L^N(\{u = T\})} = 0.$$

In particular, there exists $0 < \kappa < T$ such that

$$C\|f\|_{L^N(\{u>\kappa\})} < 1.$$

We deduce from (2.2) that

$$\|G_\kappa(u)\|_{L^{\frac{N}{N-1}}(\Omega)} \leq 0.$$

Therefore, $u \leq \kappa$ a.e. in Ω . Hence $T = \|u^+\|_{L^\infty(\Omega)} \leq \kappa$, and this contradicts the choice of κ .

We conclude that $u \leq 0$ a.e. in Ω . Since the function $-u$ satisfies the same equation with datum $-f$, we also have $-u \leq 0$ a.e. in Ω . Thus, $u = 0$ a.e. in Ω , which cannot be the case. The proof of the theorem is complete. \square

ACKNOWLEDGEMENTS

The authors would like to thank Francesco Petitta for bringing to their attention problems related to the 1-Laplacian. The second author (ACP) was supported by the Fonds de la Recherche scientifique–FNRS. He warmly thanks the Dipartimento di Matematica of Sapienza–Università di Roma for the invitation and hospitality.

REFERENCES

- [1] F. Andreu, A. Dall’Aglio, and S. Segura de León, *Bounded solutions to the 1-Laplacian equation with a critical gradient term*, *Asympt. Anal.*, to appear. $\uparrow 2$
- [2] F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón, *The Dirichlet problem for the total variation flow*, *J. Funct. Anal.* **180** (2001), 347–403. $\uparrow 2$
- [3] G. Bellettini, V. Caselles, and M. Novaga, *Explicit solutions of the eigenvalue problem $-\operatorname{div}\left(\frac{Du}{|Du|}\right) = u$ in \mathbf{R}^2* , *SIAM J. Math. Anal.* **36** (2005), 1095–1129. $\uparrow 2$
- [4] M. Cicalese and C. Trombetti, *Asymptotic behaviour of solutions to p -Laplacian equation*, *Asymptot. Anal.* **35** (2003), 27–40. $\uparrow 2$
- [5] F. Demengel, *Some existence results for noncoercive “1-Laplacian” operator*, *Asymptot. Anal.* **43** (2005), 287–322. $\uparrow 2$
- [6] J. Leray and J.-L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques non-linéaires par les méthodes de Minty-Browder*, *Bull. Soc. Math. France* **93** (1965), 97–107. $\uparrow 1$
- [7] J. M. Mazón and S. Segura de León, *The Dirichlet problem for a singular elliptic equation arising in the level set formulation of the inverse mean curvature flow*, preprint. $\uparrow 2$
- [8] A. Mercaldo, S. Segura de León, and C. Trombetti, *On the behaviour of the solutions to p -Laplacian equations as p goes to 1*, *Publ. Mat.* **52** (2008), 377–411. $\uparrow 2$
- [9] ———, *On the solutions to 1-Laplacian equation with L^1 data*, *J. Funct. Anal.* **256** (2009), 2387–2416. $\uparrow 2$
- [10] B. Kawohl, *On a family of torsional creep problems*, *J. Reine Angew. Math.* **410** (1990), 1–22. $\uparrow 2$

LUIGI ORSINA
"SAPIENZA", UNIVERSITÀ DI ROMA
DIPARTIMENTO DI MATEMATICA
P.LE A. MORO 2
00185 ROMA
ITALY

AUGUSTO C. PONCE
UNIVERSITÉ CATHOLIQUE DE LOUVAIN
INSTITUT DE RECHERCHE EN MATHÉMATIQUE ET PHYSIQUE
CHEMIN DU CYCLOTRON 2, BTE L7.01.02
1348 LOUVAIN-LA-NEUVE
BELGIUM